

Stable curved solitonic surfaces in nonholonomic frame

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

Assuming the stability of soliton surfaces of vanishing Ricci sectional curvature of soliton metric in the nonholonomic frame, we find a solution for the metric in the approximation of weak constant torsion curves with constant Frenet curvature. The computation of the Riemann tensor of the soliton metric shows that it does not vanish and therefore the solution is nontrivial. Heisenberg solitonic equation is also used to constrain the the soliton Riemann metric. The new feature here is that the coordinate curves on the soliton-like surface are composed of hydrodynamical filaments. **PACS numbers:** 02.40.Hw-Riemannian geometries

I Introduction

Recently T. Kambe [1] investigated the geometry and stability of solitons given by KdV equations of water waves as diffeomorphic flow, by computing the Riemann and sectional curvature, as well as the Killing vectors of the solitons. The stability of geodesic flows have been recently investigated by Kambe [1] by making use of the technique of Ricci sectional curvature [2], where the negative sectional curvature indicates instability of the flow, while positivity or null indicates stability. In the case of instability the geodesics deviate from the perturbation of the fluid. In this letter, the sectional Riemann curvature of the geodesic flow for a Riemannian soliton surface [3], in the of nonholonomic frame, where the curves on the soliton surface possesses Frenet curvature and torsion, and where the legs of the nonholonomic frame, depend upon other coordinate directions orthogonal to the coordinate along one of the curves. In the some approximations and assuming the stability of the soliton surface metric along with Heisenberg solitonic constraint, we find the metric for this soliton surface for incompressible filamentary flows. A distinct aspect between Kambes and ours approach is that the sectional curvature does not depend on the flow speed but just on the geometrical quantities of the flow. This is similar to an idea of Thifeault [4] which describes the Riemannian geometry of a curved substrate. The paper is organized as follows: Section II presents a brief review of Riemannian geometry in the coordinate free language. Section III presents the sectional curvature for the solitonic like surface. Section IV presents the conclusions.

II Ricci and sectional Riemann curvatures

In this section we make a brief review of the differential geometry of surfaces in coordinate-free language. The Riemann curvature is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (\text{II.1})$$

where $X \in T\mathcal{M}$ is the vector representation which is defined on the tangent space $T\mathcal{M}$ to the manifold \mathcal{M} . Here $\nabla_X Y$ represents the covariant derivative given by

$$\nabla_X Y = (X \cdot \nabla) Y \quad (\text{II.2})$$

which for the physicists is intuitive, since we are saying that we are performing derivative along the X direction. The expression $[X, Y]$ represents the commutator, which on a vector basis frame \vec{e}_l in this tangent sub-manifold defined by

$$X = X_k \vec{e}_k \quad (\text{II.3})$$

or in the dual basis ∂_k

$$X = X^k \partial_k \quad (\text{II.4})$$

can be expressed as

$$[X, Y] = (X, Y)^k \partial_k \quad (\text{II.5})$$

In this same coordinate basis now we are able to write the curvature expression (II.1) as

$$R(X, Y)Z := [R^l_{j k p} Z^j X^k Y^p] \partial_l \quad (\text{II.6})$$

where the Einstein summation convention of tensor calculus is used. The expression $R(X, Y)Y$ which we shall compute bellow is called Ricci curvature. The sectional curvature which is very useful in future computations is defined by

$$K(X, Y) := \frac{\langle R(X, Y)Y, X \rangle}{S(X, Y)} \quad (\text{II.7})$$

where $S(X, Y)$ is defined by

$$S(X, Y) := \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2 \quad (\text{II.8})$$

where the symbol \langle, \rangle implies internal product.

III Ricci stable solitonic surfaces

In this section we shall consider the metric $g(X, Y)$ of line element of solitonic surface defined as the first fundamental form of differential form[3]

$$I_{\Sigma} = ds^2 + g(s, b)db^2 \quad (\text{III.9})$$

This is a Riemannian line element

$$ds^2 = g_{ij}dx^i dx^j \quad (\text{III.10})$$

where $g_{11} = 1$ and $g_{22} = g(s, b)$. The curves or filaments generating this solitonic like surface is given by the two vector tangent fields

$$X = u_s(s)\vec{t} \quad (\text{III.11})$$

and

$$Y = u_b(s)\vec{t} \quad (\text{III.12})$$

obeying the incompressible flow equation

$$\nabla \cdot \vec{u} = \partial_s u_s + g^{-\frac{1}{2}} \partial_b u_b(s) \vec{t} = 0 \quad (\text{III.13})$$

Let us now assume stability of the filamentary flows on the soliton like surface embbeded in the Euclidean manifold \mathcal{E}^3 , and compute the Ricci sectional curvature above step by step. We need first to make use of the grad operator in the Riemannian solitonic metric, which is given by

$$\nabla = [\partial_s, g^{-\frac{1}{2}} \partial_b, \partial_n] \quad (\text{III.14})$$

The non-holonomic dynamical relations from vector analysis and differential geometry of curves [5] such composed the Frenet frame $(\vec{t}, \vec{n}, \vec{b})$ equations

$$\vec{t}' = \kappa \vec{n} \quad (\text{III.15})$$

$$\vec{n}' = -\kappa \vec{t} + \tau \vec{b} \quad (\text{III.16})$$

$$\vec{b}' = -\tau \vec{n} \quad (\text{III.17})$$

and the other frame direction legs are given by

$$\frac{\partial}{\partial n}\vec{t} = \theta_{ns}\vec{n} + [\Omega_b + \tau]\vec{b} \quad (\text{III.18})$$

$$\frac{\partial}{\partial n}\vec{n} = -\theta_{ns}\vec{t} - (\text{div}\vec{b})\vec{b} \quad (\text{III.19})$$

$$\frac{\partial}{\partial n}\vec{b} = -[\Omega_b + \tau]\vec{t} - (\text{div}\vec{b})\vec{n} \quad (\text{III.20})$$

$$\frac{\partial}{\partial b}\vec{t} = \theta_{bs}\vec{b} - [\Omega_n + \tau]\vec{n} \quad (\text{III.21})$$

$$\frac{\partial}{\partial b}\vec{n} = [\Omega_n + \tau]\vec{t} - \kappa + (\text{div}\vec{n})\vec{b} \quad (\text{III.22})$$

$$\frac{\partial}{\partial b}\vec{b} = -\theta_{bs}\vec{t} - [\kappa + (\text{div}\vec{n})]\vec{n} \quad (\text{III.23})$$

Therefore to compute the Ricci tensor step by step we start by the second term in the Ricci tensor is

$$< \nabla_X \nabla_Y Y, X > = -u_s^2 u_b^2 \kappa_0 g^{-\frac{1}{2}} [\kappa + \text{div}(\vec{n})] \quad (\text{III.24})$$

where we have used the helical filaments hypothesis $\kappa_0 = \text{constant} = \tau_0$. The other terms in the Ricci tensor are

$$[X.Y] = -u_s u_b [1 - g^{-\frac{1}{2}}] \vec{n} \quad (\text{III.25})$$

which implies that

$$\nabla_{[X,Y]} Y = -u_s u_b [1 - g^{-\frac{1}{2}}] \vec{n} \quad (\text{III.26})$$

In these last equations we have used the Heisenberg constraint equation [5]

$$\partial_s \kappa = \theta_{bs} \kappa \quad (\text{III.27})$$

which allows us to place $\theta_{bs} = 0$ in future computations.

The sectional curvature is thus

$$K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{S(X, Y)} = -[1 - g^{-\frac{1}{2}}]\theta_{ns} + \tau_0^2 - g^{-\frac{1}{2}}[\tau_0^2 + \tau_0 \text{div}\vec{n}] \quad (\text{III.28})$$

Since it is assumed that the solitonic surface is stable we simply place $K(X, Y) = 0$ in the last expression, which yields the following solution for the solitonic metric

$$g(s, b) = [1 + \frac{\text{div}\vec{n}}{\tau_0}]^2 \quad (\text{III.29})$$

when the filaments possess a very weak torsion we can approximate this expression by

$$g(s, b) \approx [\frac{\text{div}\vec{n}}{\tau_0}]^2 \quad (\text{III.30})$$

Note that when both angular velocity and perturbation both keep the same sign, the sectional curvature $K(X, Y)$ is negative and the flow along the Ricci soliton is stable.

IV Conclusions

An important issue in plasma astrophysics as well as in fluid mechanics and optics is to know when a fluid, charged or not, is unstable or not. In this we invert the problem and assume stability of solitonic-like surfaces and found with the appropriated constraints the metric for this soliton surface. Incompressible flows filaments are used in the formation of the curved surface.

References

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